Example Induction Proofs on Trees

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Induction proof on trees

Claim: In a binary tree of height h, the number of nodes $n \leq 2^{h+1} - 1$.

Proof: Use Induction on height of tree.

Base case: $h = 0$

$2^{h+1} - 1 = 2^1 - 1 = 1$

$n = 1$

$\therefore \quad n \leq 2^{h+1} - 1$

(contd.)

$T \quad h = 0$

Tree can only have one node

overhead

Claim: In a binary tree of height h, the number of nodes $n \leq 2^{h+1} - 1$.

Proof (continued):

Induction Hypothesis: $\forall h \leq k, n \leq 2^{h+1} - 1$.

Want to prove: For $h = k + 1, n \leq 2^{h+2} - 1$.

***Induction* step**: Recall that in each node in a binary tree has at most two children.

Consider a tree with height $k + 1$ with n nodes. Let's call its root node r_0 . There are two possibilities: r_0 has one child, or r_0 has two children.

Case 1: r_0 has (exactly) one child. Call the child r_1.

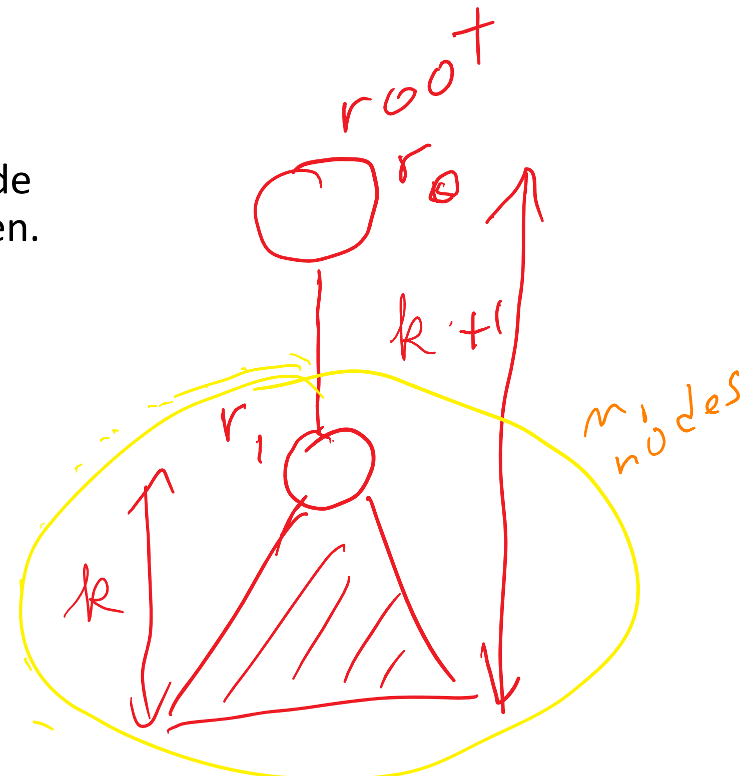
Suppose the subtree with root r_1 has n_1 nodes. Then $n = n_1$+1.

Now the height of the subtree rooted at r_1 must be k (otherwise the height of the entire tree rooted at r_0 could not be $k + 1$).

By induction hypothesis: $n_1 \leq 2^{k+1} - 1$. Therefore $n \leq 2^{k+1}$.

Now $2^{k+1} \leq 2^{k+2} - 1$ for all $k \geq 0$, thus $n \leq 2^{h+2} - 1$.

We proceed to Case 2.



Claim: In a binary tree of height h, the number of nodes $n \leq 2^{h+1} - 1$.
Proof (continued):
Induction Hypothesis: $\forall h \leq k, n \leq 2^{h+1} - 1$.
Want to prove: For $h = k + 1, n \leq 2^{k+2} - 1$.

Proof (continued). ***Case 2***: Suppose tree rooted at r_0 has height k+1 and (exactly) two children. Call the children r_1 and r_2.

Suppose the subtrees rooted at r_1 and r_2 have n_1 and n_2 nodes, and heights k_1 and k_2, respectively.

Now $n = n_1 + n_2 + 1$, $k = \max(k_1, k_2) + 1$, $k_1 \leq k$, and $k_2 \leq k$.
Without loss of generality suppose $k_1 \leq k_2$. This implies $k = k_2$.

By induction hypothesis:

$n_1 \leq 2^{k_1+1} - 1$ and $n_2 \leq 2^{k_2+1} - 1$

Thus $n \leq (2^{k_1+1}-1) + (2^{k_2+1}-1) + 1$,

i.e. $n \leq 2^{k_2+1} + 2^{k_2+1} - 1$. So $n \leq 2^{k_2+2} - 1$. But $k = k_2$.

This implies $n \leq 2^{k+2} - 1$.

Which is what we wanted to prove. QED.